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The Mach wave field radiated by supersonic turbulent shear flows

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Theoretical studies of aerodynamic noise suggest that the sound field of supersonic flows will be dominated by eddy Mach waves. Recent experimental evidence supports this view. In supersonic turbulent boundary layers, and rocket exhaust flows, turbulence occurs in regions of high mean velocity gradient. At low speed, such gradients are known to amplify the sound emitted by turbulence. This paper deals with the corresponding Mach wave problem. The exact equations of sound radiation by turbulence are rearranged in a form where the equivalent sources, derivatives of the turbulence stress tensor, are shown to be dominated by one term. That term is formed from the product of the mean velocity gradient and the rate of change of density. It seems that its resemblance to the dominant source of sound in low speed shear flows is largely fortuitous. In the Mach wave case, the theory is designed to include effects of both temperature gradients and density perturbations, and the approximations of the estimate are of a type that would not be expected to be valid away from the Mach wave condition. The basic theory is used to make an estimate of the sound radiated from supersonic boundary layers, and an approximate equation relating the radiated pressure to the surface pressure is derived. Experimental evidence is then examined to show that the equation is in excellent agreement with observation. The theory is then applied to annular shear flows of the rocket exhaust type. Again an approximate equation relating near and far field pressures is established, and the paper concludes with suggestions for experiments that could check the result.

1. Introduction

Turbulent airflow is known to radiate sound with increasing efficiency at high speeds. That increase is due to three effects. At low speeds, the most important of these are the rise in turbulence levels and frequencies with increasing flow velocity. But the other effect, though small at low speeds, is one that completely changes the character of radiation from supersonic flow, and is due to the convective motion of the coherent regions of turbulence that we call eddies. It is only to be expected that eddies moving supersonically should create rudimentary shock waves. Such waves are highly directional, having their fronts aligned to the flow at, or near, the Mach angle, a property that led Phillips (1960), who was the first to emphasize that feature, to term them Mach waves. It is now well known that Mach waves dominate the radiation field or supersonic turbulent flow (Laufer 1961), so that their study is relevant to most problems of sound generation by turbulence at high speed. Their study in flows of the type that occur in rocket exhausts is particularly relevant, for it may help to throw important new light on the still open question regarding the location of the main noise-producing regions of those flows.

Lighthill's (1952, 1954) theory of aerodynamic noise provides a basis for the study of sound induced aerodynamically. There the radiation field is defined explicitly provided that a turbulence stress tensor is known throughout the flow. If velocities are low enough for temperature and density changes within the turbulence to be negligible, the stress tensor approximates to the fluctuating Reynolds stress in an incompressible fluid. Incompressible flow arguments can then serve to estimate the strength of the equivalent aerodynamic quadrupoles which are the source terms of Lighthill's equations. Proudman (1952) was able to exploit this technique in his theory of sound radiated by isotropic turbulence. At high speeds the situation is more complicated. Although Lighthill's theory is exact and has been shown to form a consistent analytical basis for the study of sound generated by turbulent flow, up to high supersonic speeds (Ffowcs Williams 1963) it pre-supposes a knowledge of the stress tensor, which is itself subject to compressibility effects. In fact a proper treatment of any aspect of the sound-flow interaction problem, which inevitably assumes a crucial role at high Mach number, requires an admission that the stress tensor includes terms directly attributable to the sound field. The stress tensor cannot then be estimated until the radiation problem is solved and Lighthill's solution appears as a highly intractable integral equation. Furthermore, at high Mach numbers the temperature is likely to be subject to kinetic heating effects and can no longer be regarded as constant; an essential assumption in approximating the stress tensor by the Reynolds stress.

True though these arguments are, they present an unduly depressing outlook on the prospects of the theory at high Mach numbers. This is particularly the case if we accept that the main purpose of theory must be to estimate the radiation field induced by some known property of the turbulent flow. That known property may be deduced from analytical studies, measured by direct experiment or, as is more likely to be the case, estimated very crudely by dimensional and similarity arguments. It is inevitable that different theoretical approaches will be required to yield solutions based on different descriptions of the turbulent flow, and that many problems will defy realistic solutions. However, there remain instances when model flows can be constructed that are both realistic and tractable, and the case of Mach wave generation by highly sheared supersonic flow seems to be one of these. But even then, much depends on what particular aspect of the turbulent flow one assumes to be known.

Phillips (1960) was the first to make a thorough attack on the supersonic shear flow problem. His theory was designed to yield a description of the Mach wave field based on a known distribution of turbulent velocities. This aim necessitated a major reworking of the basic theory which Phillips achieved through an elegant manipulation of the equations governing the motion of an ideal gas. He thus derived a convected wave equation forced by a term that could be described closely in terms of the known velocity field. The solution of that equation presents a formidable challenge, and the Phillips (1960) technique was to seek an asymptotic expansion at high Mach numbers, for a model shear flow. More general solutions have yet to be worked out, but his approach forms the basis for estimating the radiation field once the turbulent velocities are known.

However, if, in place of the velocity field, the turbulence were described by some product of density and velocity, such as may be obtained by hot-wire studies in the turbulent flow, the Mach wave field would have tò be tackled by a different approach. Perhaps Lighthill's stress tensor could then be estimated more readily than could the velocity field, so that his equations would form a more direct procedure for solving the radiation problem. This point will not be emphasized further. It is introduced merely as a reminder that both the Phillips and Lighthill starting equations are exact and either could form a sound basis for estimating the radiation field, provided the respective forcing fields were adequately described. Different models of the flow may clearly favour different approaches so that the two theories should be regarded as complimentary.

In this paper we develop a technique for estimating the Mach wave radiation from a supersonic turbulent shear flow based on a description of the pressure or density fluctuations within that flow. It is well known that the perturbation pressure field is relatively extensive in comparison to the turbulent velocity scales so that one may be able to estimate the order of magnitude of that pressure from a straightforward experimental survey in the near vicinity of the flow. The aim of our theory is then to relate the Mach wave field to a turbulent property that might be estimated, however crudely, by direct experiment. It transpires that for the particular case of Mach wave emission in a highly sheared flow, the source term of an exact equation based on a development of Lighthill's theory, is approximated to high accuracy, by a comparatively simple expression involving pressure as the only unsteady parameter. In that development, account is taken of both density and temperature changes so that the theory makes some attempt at overcoming the pitfalls of the usual approximations that feature in aerodynamic noise computations.

The theory is applied to the radiation field of turbulent boundary layers, where an estimate of Mach wave strength is based on measurements of the surface pressure. The agreement between the theoretical computation and experimentally measured values is remarkably good, both as regards the Mach number dependence and the order of magnitude. The paper concludes with an application of the same technique to a model flow of the rocket exhaust type. The Mach wave field is estimated in terms of the local pressure levels, and experiments that could throw light on the relevance of the theory to practical rocket noise problems are suggested.

2. Mach wave sources in supersonic turbulent shear layers

Within the framework of Lighthill's (1952) acoustic analogy, turbulent flow radiates sound to distant points in quiescent air in an equivalent way to that radiated by quadrupoles in a homogeneous medium at rest. The quadrupole strength is equal to a turbulence stress tensor which is generally assumed known. Convective motion has a pronounced effect on the radiation efficiency of quadrupole sources, for they radiate more effectively at high frequencies. The Doppler effect increases frequency in the direction of motion, an increase that accounts for the major convective amplification of the radiated sound. But whenever sources approach an observer with precisely the speed of sound, the Doppler factor becomes singular and the radiation problem undergoes a drastic change of character, the rather inefficient quadrupole radiation giving way to a highly directional intense wave system known as Mach waves (Phillips 1960). It is with the Mach waves generated by high-speed shear flows that we are currently concerned, a problem pertinent to the study of both rocket noise and the sound of high-speed turbulent boundary layers.

Aerodynamically, the Mach wave concept is perfectly clear, being a form of ballistic shock wave attached to coherent regions of turbulence, or eddies, convected supersonically. But it is within the aerodynamic noise theory that their study is tractable, and there they require a different but equally revealing interpretation. A convected eddy is acoustically equivalent to a moving quadrupole, essentially of low radiation efficiency, composed, as it is, of four mutually opposed simple sources. No radiation would result if these sources were heard simultaneously. However, retarded time effects destroy the coherence, so that cancellation is rarely complete. When a quadrupole moves towards an observer, the nearer elements emit from a relatively closer position, for they have moved forward in the retarded time interval separating emission from the near and far regions. The quadrupole then appears to occupy a larger volume, and the retarded time interval is increased, bringing about a reduced cancellation which results in increased emission.

Obviously, this increase cannot be maintained indefinitely, for a quadrupole could at no time radiate more effectively than its constituent elementary sources (Lighthill 1962), a condition corresponding to a complete absence of cancellation. Such is the case when a quadrupole approaches an observer at the speed of sound. Then the waves emitted by the more distant elements never overtake the nearer portions, and each element is heard independently. This is the Mach wave phenomenon where the quadrupoles degenerate into their constituent elementary sources (Ffowcs Williams 1963). An eddy in supersonic motion is thus acoustically equivalent to a quadrupole, whose radiation is enhanced in accordance with the Stokes and Doppler effects, except in the one direction where it travels at the speed of sound. That is the direction in which Mach waves propagate, and their strength exceeds those of other waves by the ratio of simple source to quadrupole efficiency, generally a very large factor. This physical introduction helps to make clear the sometimes subtle analytical manipulations of aerodynamic noise theory, to which we will now turn, to consider more specifically the Mach waves generated by supersonic shear flows.

The acoustic analogy starts with a rearrangement of the exact equations of fluid motion as an inhomogeneous wave equation for density ρ,

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{2.1}$$

where T_{ij} is Lighthill's turbulence stress tensor, $\rho u_i u_j + p_{ij} - a_0^2 \rho \delta_{ij}$, u_i being the velocity in the i direction, a_0 the speed of sound in the uniform medium surrounding the turbulent flow, δ_{ij} the Kronecker delta and p_{ij} a tensor incorporating both pressure and viscous terms. Repeated tensor suffices are to be summed over 1, 2 and 3.

In the absence of solid surfaces equation (2.1) can be rewritten to yield a solution for ρ once the turbulence stress tensor is known,

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^2}\int_v \left[\frac{\partial^2 T_{ij}}{\partial y_i\,\partial y_j}\right]\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{2.2}$$

This integral should be evaluated over all space, but it is only from the turbulent region that it receives significant contributions. In reality, equation (2.2) is a rather complicated and intractable integral equation for density, T_{ij} itself being a function of density. But we shall not dwell on that point. Rather, we argue below that T_{ij} can be approximated, to high accuracy, by a relatively simple term, in this instance of high shear flow turbulence. Equation (2.2) can then remain, as it has always been regarded at low Mach numbers, an expression that the sound is generated by a distributed system of aerodynamic sources of known strength.

The double divergence of the stress tensor is the simple source strength density, which integrates instantaneously to zero. This point is a more precise expression of our foregoing discussion, that simple sources are arranged in opposing sets that constitute quadrupoles. The mutual cancellation makes difficult an assessment of the radiated sound based on equation (2.2), and Lighthill showed how the quadrupole character could be brought to light by applying the divergence theorem twice and disregarding surface integrals. His equation can be expressed in its far field form as

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^4}\int_v \frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\frac{\partial^2 T_{ij}}{\partial t^2}\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)d\mathbf{y}. \tag{2.3}$$

This equation appears more straightforward when rewritten in a terminology first introduced by Proudman (1952) where the tensor suffices, i and j, are replaced by r, a notation implying the direction of emission and requiring no summation of repeated suffices

$$\{\rho-\rho_0\}(\mathbf{x},t) = \frac{1}{4\pi a_0^4}\int_v \frac{\partial^2 T_{rr}}{\partial t^2}\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{2.4}$$

The convective effects can now be given a more rigorous basis. Whenever eddies move downstream, part of the time derivative $\partial T_{rr}/\partial t$ is generated by the convection of a spatial gradient, $\partial T_{rr}/\partial y$. If the convective speed is $a_0 M$, $\partial T_{rr}/\partial t$ is approximately equal to $a_0 M(\partial T_{rr}/\partial y)$. Such a space derivative integrates instantaneously to zero, and thus represents a source of basically higher order and of fundamentally lower efficiency. Lighthill (1952) chose a system of axes attached to the eddies, $\boldsymbol{\eta} = \mathbf{y}-a_0\mathbf{M}t$, to illustrate this effect and his result is the basis of aerodynamic noise theory at significant Mach number

$$\{\rho-\rho_0)(\mathbf{x},t) = \frac{1}{4\pi a_0^4}\int_v \frac{\partial^2 T_{rr}}{\partial t^2}\left(\boldsymbol{\eta}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{|\mathbf{x}-\mathbf{y}|^2\,d\boldsymbol{\eta}}{[|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})]^3} \tag{2.5}$$

$$= \frac{1}{4\pi a_0^4}\int_v \frac{\partial^2 T_{rr}}{\partial t^2}\left(\boldsymbol{\eta}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\boldsymbol{\eta}}{|\mathbf{x}-\mathbf{y}|\,(1-M\cos\theta)^3}. \tag{2.6}$$

The apparent singularity at $M\cos\theta = 1$ (θ being the radiation angle measured from the direction of convective motion) heralds the powerful and highly directional Mach wave radiation. Then, in accordance with our physical arguments (as well as more general theoretical concepts, Ffowcs Williams 1963) the simple source strength is the relevant measure of radiation efficiency, and we must revert to equations (2.2) or (2.4), discarded above as not revealing of the more classical radiation of convected quadrupoles. However, they are both precise, and it is on them that we base our analysis. Our starting equation will be an intermediate stage, where only one of the divergences of equation (2.2) has been replaced by a time derivative. That is chosen for analytical convenience that will become apparent in our manipulation of the source term

$$\{\rho-\rho_0\}(\mathbf{x},t) = -\frac{1}{4\pi a_0^3}\int_v \left[\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i}\right]\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}; \tag{2.7}$$

$1/a_0(\partial^2 T_{ri}/\partial t\,\partial y_i)$ is now a measure of the equivalent aerodynamic source strength and assumes a particularly simple form for Mach waves generated by a highly sheared flow. That form we now derive by employing the equations of motion to emphasize that part of the source term directly amplified by a mean velocity gradient

$$\frac{1}{a_0}\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i} = \frac{1}{a_0}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial y_i}(\rho u_r u_i + p_{ri}) - a_0^2\frac{\partial \rho}{\partial y_r}\right\}. \tag{2.8}$$

The momentum equation allows this to be rewritten

$$\frac{1}{a_0}\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i} = -\frac{1}{a_0}\frac{\partial}{\partial t}\left\{\frac{\partial(\rho u_r)}{\partial t} + a_0^2\frac{\partial \rho}{\partial y_r}\right\}. \tag{2.9}$$

A comparison of equations (2.2) and (2.4) will reveal an integral equivalence of derivatives with respect to time and a divergence of a vector field. We now make use of this equivalence in noting that a source of strength $\partial(\rho u_i)/\partial y_i$ is precisely equivalent to a source of strength $-a_0^{-1}\,\partial(\rho u_r)/\partial t$; i.e. they both generate identical sound fields far away from the source region. Furthermore, gradients in directions normal to the radiation direction integrate directly to zero, the integral being an instantaneous one involving no retarded time change, so that $\partial(\rho u_i)/\partial y_i$ is precisely equivalent to $\partial(\rho u_r)/\partial y_r$ in generating sound. This, in turn, is equivalent to $-a_0^{-1}\,\partial(\rho u_r)/\partial t$ so that the source term of equation (2.9) can be written in its fully equivalent form

$$\frac{1}{a_0}\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i}\int = -\frac{1}{a_0}\frac{\partial}{\partial t}\left\{a_0^2\frac{\partial \rho}{\partial y_r} - a_0\frac{\partial(\rho u_r)}{\partial y_r}\right\}. \tag{2.10}$$

The symbol $\int =$ implies that the two sides of the equation integrate at retarded time to yield identical distant sound fields, which is our current concern, but does not imply a direct equality of functions. The right-hand side of equation (2.10) can thus be regarded as the source term and may be expanded to the form

$$\frac{1}{a_0}\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i}\int = -\frac{\partial}{\partial t}\left\{(a_0-u_r)\frac{\partial \rho}{\partial y_r} - \rho\frac{\partial u_r}{\partial y_r}\right\}. \tag{2.11}$$

We seek the particular term subject to amplification by a mean velocity gradient, so we consider the mean value of u_r, $\bar{u}_r$, separately from its time-dependent fluctuation, u'_r. At this stage the equations remain exact, so that the Mach wave source strength is given precisely by

$$\frac{1}{a_0}\frac{\partial^2 T_{ri}}{\partial t\,\partial y_i}\int = (\bar{u}_r - a_0)\frac{\partial^2 \rho}{\partial y_r\,\partial t} + \frac{\partial \rho}{\partial t}\frac{\partial \bar{u}_r}{\partial y_r} + \frac{\partial^2(\rho u'_r)}{\partial y_r\,\partial t}. \tag{2.12}$$

Now we suppose the mean velocity gradients to be sufficiently high that the second term on the right-hand side of this equation, the only one directly amplified by such a gradient, dominates the source system, so that the remaining terms are negligible. This supposition would appear highly relevant to the practical situation, since both the remaining terms are likely to be of little importance. The first is small since the centre of an eddy will move at a speed close to the mean flow velocity, so that $\bar{u}_r$ is approximately the component of convection speed in the direction of emission, by definition equal to a_0 for the Mach wave problem. In a high-velocity gradient the mean velocity changes rapidly, but if density remains correlated in a symmetrical way, the term will tend to be antisymmetric about the eddy centre and will therefore tend to vanish on integration. The remaining term involves the density and perturbation velocity only, and must, if the turbulence level remains relatively low, be small compared to the second term, particularly if the mean flow gradients greatly exceed the fluctuating velocity gradients, which we assumed to be the case.

We express the mean velocity gradient, $\partial\bar{u}_r/\partial y_r$, in its tensor form

$$(x_i - y_i)(x_j - y_j)/|\mathbf{x}-\mathbf{y}|^2\,(\partial\bar{u}_i/\partial y_j),$$

and assume a local isentropic condition, so that $\partial\rho/\partial t$ is replaced by $a^{-2}(\partial p/\partial t)$, in writing our basic equation describing Mach waves radiated by a highly supersonic shear flow

$$\{\rho - \rho_0\}(\mathbf{x}, t) = -\frac{1}{4\pi a_0^2}\int_v \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3}\left[\frac{1}{a^2}\frac{\partial p}{\partial t}\frac{\partial \bar{u}_i}{\partial y_j}\right]\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)d\mathbf{y}. \tag{2.13}$$

Though this equation is of the form suggested by Lighthill (1954) to be relevant to sound production by low-speed shear layers, the agreement seems largely fortuitous. Close examination reveals that the radiation is precisely opposite in sign, and that the equations differ by a factor $2(a^2/a_0^2)$. But a substantial difference would be expected, for the low-speed result, based on the neglect of octupole terms, must vary considerably from the Mach wave case where each multipole assumes a simple source efficiency. The remarkable aspect is rather that the low-speed result is at all similar in form, so the similarity will not be considered further.

Equation (2.13) is the main result of this paper and will form the basis of our estimate of the sound radiated by supersonic shear layers. That estimate is based on idealized models of the turbulent flow, but models that have some experimental foundation, at least at low speeds.

3. Mach wave radiation from supersonic boundary layers

The Mach waves radiated by supersonic turbulent boundary layers provide an instance where the foregoing theory should be of direct bearing. Their experimental investigation, recently reported by Laufer (1961), provides a basis for a critical assessment of our theoretical model, by comparison of predicted and measured features of the radiation field. This is what we now set out to do. We regard the boundary layer to have uniform properties over an infinite plane surface. This step allows us to disregard any influence of the boundary in our computation of the sound field. In fact, we assume viscous effects to be negligible (except, of course, that they play their full role in maintaining the turbulent state), so that Powell's (1960) argument, that a plane boundary merely reflects the aerodynamically generated sound, is relevant. We restrict our attention entirely to Mach waves, and specifically to the shear amplified waves with their characteristic directionality. This step, quite apart from directing our attention to those waves of greatest significance, avoids any trouble with singularities in the nature of Olbers' paradox usually present whenever sources are distributed over an infinite area.

We choose our co-ordinate system such that the mean flow is in the 1-direction. The 2-direction is that normal to the boundary surface, so that the only mean shear is $\partial \bar{u}_1/\partial y_2$. We assume adiabatic conditions in the radiation field and rewrite the mean square pressure, $\overline{p^2}(\mathbf{x})$, as equal to $\overline{a_0^4(\rho-\rho_0)^2}(\mathbf{x})$. This pressure is given by the mean square value of equation (2.13). We write that equation for the particular case of small eddies centred at $\mathbf{y}$, and assume that both the velocity gradient $\partial \bar{u}_1/\partial y_2$ and the local speed of sound a remain constant over a distance of the order of an eddy length. Consequently, they may be regarded as functions only of the eddy location.

$$\overline{p^2}(\mathbf{x}) = \frac{1}{16\pi^2}\int_v \frac{(x_1-y_1)^2(x_2-y_2)^2}{|\mathbf{x}-\mathbf{y}|^6}\left\{\frac{1}{a^2}\frac{\partial \bar{u}_1}{\partial y_2}\right\}^2 (y_2) \times \int_v \overline{\frac{\partial p}{\partial t}\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{\partial p}{\partial t}\left(\mathbf{y}+\boldsymbol{\lambda}, t-\frac{|\mathbf{x}-\mathbf{y}-\boldsymbol{\lambda}|}{a_0}\right)}\, d\boldsymbol{\lambda}\, d\mathbf{y}. \quad (3.1)$$

It was shown by Ffowcs Williams (1963) that the integration over the correlation volume should be replaced by an integration over the correlation area normal to the radiation direction in association with an integration over the moving axis time scale, whenever Mach waves are under study. We carry out this step. We replace the integral of the correlation function by the mean square value of the pressure time derivative, $\overline{(\partial p/\partial t)^2}(y_2)$, multiplied by the correlation volume, in this case equal to $a_0\tau^*\lambda_s^*$, $a_0\tau^*$ being the distance travelled by an eddy in the direction of emission during its lifetime τ^*, and λ_s^* being the correlation area in the Mach wave plane

$$\int_v \overline{\frac{\partial p}{\partial t}\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{\partial p}{\partial t}\left(\mathbf{y}+\boldsymbol{\lambda}, t-\frac{|\mathbf{x}-\mathbf{y}-\boldsymbol{\lambda}|}{a_0}\right)}\, d\boldsymbol{\lambda} = \overline{\left(\frac{\partial p}{\partial t}\right)^2}(y_2)\, a_0\tau^*\lambda_s^*. \quad (3.2)$$

The equation for the radiated pressure field then becomes

$$\overline{p^2}(\mathbf{x}) = \frac{1}{16\pi^2}\int_v \frac{(x_1-y_1)^2\,(x_2-y_2)^2}{|\mathbf{x}-\mathbf{y}|^6}\left\{\frac{1}{a_2}\frac{\partial \overline{u}_1}{\partial y_2}\right\}^2 (y_2)\,\overline{\left(\frac{\partial p}{\partial t}\right)^2}\,(y_2)\,a_0\tau^*\lambda_s^*\,d\mathbf{y}. \quad (3.3)$$

The condition for Mach wave generation imposes a geometrical constraint that requires attention be concentrated on a particular region at any one speed. Mach waves radiate in directions close to the Mach angle, an angle that varies from directly downstream to the normal direction as Mach number rises from one to infinity. At any position, height h above the boundary layer (assumed to be much thinner than h), Mach waves generated by eddies moving at near sonic speed will come from far upstream, for they travel in a near tangential direction to the boundary. On the other hand eddies moving at high supersonic speed will radiate to the point from a much closer position, so that those waves arrive with greater relative strength. Before we can integrate over any surface parallel to the plane boundary, and so sum the contribution made by sources in that surface, we must first determine how effectively the eddies radiate to angles slightly removed from the Mach angle. This we can do by reference to a particular example worked by Ffowcs Williams (1963) together with a more general dimensional argument (Ffowcs Williams 1962) that shows the radiation from convected turbulence to be proportional to the factor

$$F(M,\theta) = \left\{\frac{\epsilon^2 M^2}{(1-M\cos\theta)^2+\epsilon^2M^2}\right\}^{\frac{5}{2}}, \quad (3.4)$$

where ϵ is a coefficient that relates the typical frequency in a frame of reference convected with the turbulence to that in a fixed reference system. It is of the order of the normalized mean square turbulence level, typically near 0·2 in subsonic jet mixing flows, a value we assume pertinent to the supersonic boundary layer. At the Mach angle the coefficient $F(M,\theta)$ is seen to be unity, while it is a function of Mach number away from that region. This point reflects the fact that the simple-source-like Mach wave radiation occurs near the Mach angle and that radiation to other directions is quadrupole in nature and less efficient by a factor dependent on the convective speed. Our estimate of the Mach wave strength, together with this directional term, can thus be used to compute the total radiation, by simply weighting the integrand of equation (3.3) with the directional function $F(M,\theta)$

$$\overline{p^2}(\mathbf{x}) = \frac{1}{16\pi^2}\int_v F(M,\theta)\,\frac{(x_1-y_1)^2\,(x_2-y_2)^2}{|\mathbf{x}-\mathbf{y}|^6}\left\{\frac{1}{a^2}\frac{\partial \overline{u}_1}{\partial y_2}\right\}^2 (y_2)\,\overline{\left(\frac{\partial p}{\partial t}\right)^2}\,(y_2)\,a_0\tau^*\lambda_s^*\,d\mathbf{y}. \quad (3.5)$$

As before, M is the eddy convection speed normalized with respect to the sonic speed in the uniform flow, a_0, and $\cos\theta$ is the ratio, $(x_1-y_1)/|\mathbf{x}-\mathbf{y}|$.

The integral over planes parallel to the surface is not straightforward but can be evaluated by making use of a Taylor expansion about the point, $M\cos\theta = 1$, of the terms that weight the function $F(M,\theta)$. The series is rapidly convergent at the higher Mach numbers, the second term being smaller than the leading term by a factor $\frac{3}{4}\epsilon^4M^8(M^2-1)^{-4}$. However, the error increases rapidly near sonic convection speeds where this result takes no account of the fact that the highly directional Mach wave radiation gives way to the more nearly omnidirectional low speed result, the nature of our approximation being such as to neglect the sound radiation by subsonic flows. But that need cause no great concern since the error is small, being less than approximately 10 % at a Mach number of $(1-\epsilon)^{-1}$, equal to 1·25 when ϵ is 0·2, and decreasing rapidly at higher speeds. The leading term of the high Mach number expansion approaches zero at sonic speeds, and in selecting it we neglect the sound generated by eddies convected at Mach numbers essentially lower than $(1-\epsilon)^{-1}$, an approximation that is entirely compatible with our discussion of the Mach wave radiation. That leading term is

$$\int_{S(y_2=\text{const.})} F(M,\theta)\frac{(x_1-y_1)^2(x_2-y_2)^2}{|\mathbf{x}-\mathbf{y}|^6}\,d\mathbf{y}\,(y_1,y_3) = \frac{4\epsilon}{3}\frac{(M^2-1)^{\frac{1}{2}}}{M^3}. \tag{3.6}$$

This value, when inserted into equation (3.5) effects a considerable simplification of the radiation equation

$$\overline{p^2}(\mathbf{x}) = \frac{\epsilon}{12\pi^2}\int\frac{(M^2-1)^{\frac{1}{2}}}{M^3}\left\{\frac{1}{a^2}\frac{\partial\overline{u}_1}{\partial y_2}\right\}^2(y_2)\,\overline{\left(\frac{\partial p}{\partial t}\right)^2}(y_2)\,a_0\tau^*\lambda_s^*\,dy_2. \tag{3.7}$$

This result is independent of the observation height above the plane, in accordance with the experimental results of Laufer (1962) and the concept of energy conservation in the radiation field.

We assume now that each eddy moves downstream with a convection speed equal to the mean flow velocity at its centre, $\overline{u}_1(y_2)$. The mean velocity gradient can then be used to transform the integration over the boundary-layer thickness, to an integration over convection Mach number, $M = \overline{u}_1/a_0$. Since we restrict our attention to Mach waves, which can only be generated by eddies moving supersonically with respect to the uniform flow, the integration range is from $M = 1$, to $M = M_\infty$, M_∞ being the mean flow Mach number which is the speed at which the eddies at the boundary surface move relative to the free flow

$$\int\frac{\partial\overline{u}_1}{\partial y_2}dy_2 \to a_0\int_1^{M_\infty} dM. \tag{3.8}$$

The correlation scales must now be estimated. We assume that the moving axis time scale, or eddy lifetime, τ^*, is inversely proportional to the local mean velocity gradient. Davies, Fisher & Barratt (1963) found this to be the case for the turbulence in a jet mixing region at subsonic speeds. We shall assume, for lack of a better guide, that this property holds over into the supersonic régime and regard the product $(\partial\overline{u}_1/\partial y_2)\,\tau^*$, as a constant close to five, the value observed in subsonic jet flows.

It is of interest to study the effect of one other assumption based on flow observations at low speed (Favre, Gaviglio & Dumas 1958). That is that the eddies are elongated in the downstream direction and that their scale increases in direct proportion to the boundary-layer displacement thickness δ_1. The area λ_s^* is an oblique cross-section of an eddy, so that it increases with increasing speed, being equal to $\delta_1^2/\cos\theta$, or $\delta_1^2 M$ at the Mach wave condition.

Equation (3.7) can then be written in the more tractable form

$$\overline{p^2}(\mathbf{x}) = \frac{5\epsilon}{12\pi^2}\int_1^{M_\infty} \frac{(M^2-1)^{\frac{1}{2}}}{M^n}\frac{a_0^2}{a^4}\overline{\left(\frac{\partial p}{\partial t}\right)^2}\delta_1^2\, dM, \tag{3.9}$$

where n is either 3 or 2 depending on whether or not a correction is made for eddy elongation effects.

The mean square value of the pressure time derivative can be estimated to be of the order of the mean square pressure $\overline{p^2}$ times the square of a characteristic frequency. We shall assume that the order of magnitude of the mean square pressure is that measured by Kistler & Chen (1963) on the boundary surface, in

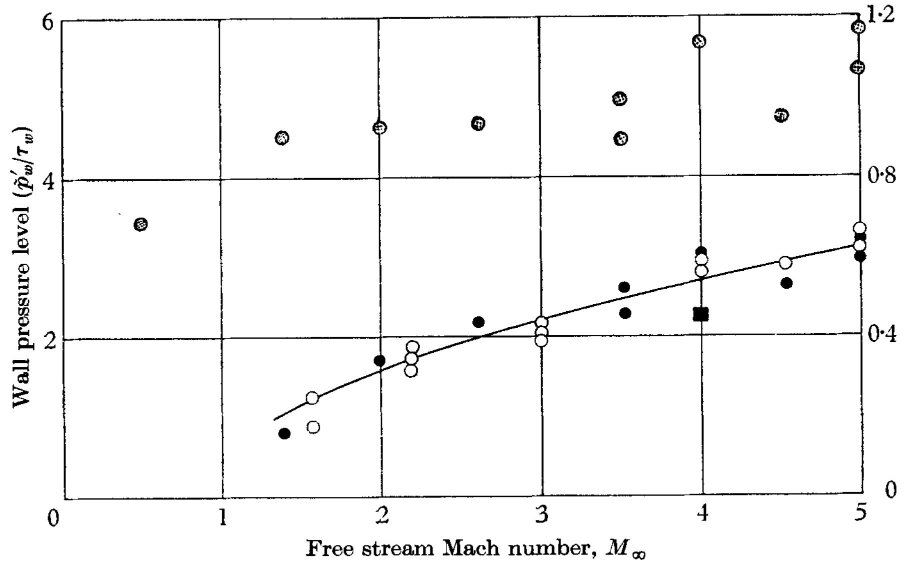

FIGURE 1. A comparison of the Mach wave strength predicted by the present theory with Laufer's experimental results (no eddy elongation). ■, Radiated pressure* (single wall configuration); ○ $\tilde{p}'/\tau_w$ radiated pressure (four-wall configuration) (Laufer 1964); ● Mach wave strength predicted by present theory; ⊛ $\tilde{p}_w'/\tau_w$ wall pressure (Kistler & Chen 1963).

the same apparatus as that used by Laufer (1962), whose results will be compared with those of this computation. Kistler & Chen found the spectrum to have its maximum at a Strouhal number near 0·3, so that we estimate $\overline{(\partial p/\partial t)^2}$ to be of the order

$$\overline{(\partial p/\partial t)^2} \simeq 0{\cdot}5|(a_0^2 M^2/\delta_1^2)\overline{p_w^2}, \tag{3.10}$$

where $\overline{p_w^2}$ is the mean square pressure at the surface. This estimate is based on the Kistler & Chen velocity measurements near the centre of the boundary layer. We have adapted their result to a reference system moving with the mean flow. The transformation rests on the hypothesis that turbulence scales remain constant across the layer and that frequencies vary according to Taylor's hypothesis of rigid convection.

The local speed of sound will be augmented due to the temperature build up caused by bringing the flow to rest near the boundary. We assume this process to be adiabatic, so that

$$a^2/a_0^2 = 1+\tfrac{1}{2}(\gamma-1)\,M^2 = 1+0{\cdot}2M^2. \tag{3.11}$$

Our final equation for the radiated sound pressure can then be written down. We normalize the values by the wall shear stress τ_w since that is the parameter used in presenting the experimental data on which we base our computation

$$\frac{\tilde{p}'(\mathbf{x})}{\tau_w} = \frac{\tilde{p}'_w}{\tau_w}\left\{\frac{5\epsilon}{24\pi^2}\int_1^{M_\infty}\frac{(M^2-1)^{\frac{1}{2}}}{M^m\{1+0{\cdot}2M^2\}^2}\,dM\right\}^{\frac{1}{2}}, \tag{3.12}$$

where $\tilde{p}'$ and $\tilde{p}'_w$ are respectively the root-mean-square Mach wave strength and the wall pressure level. m has the value 1 if we neglect the possibility of the eddy being elongated but zero if that effect is taken into account.

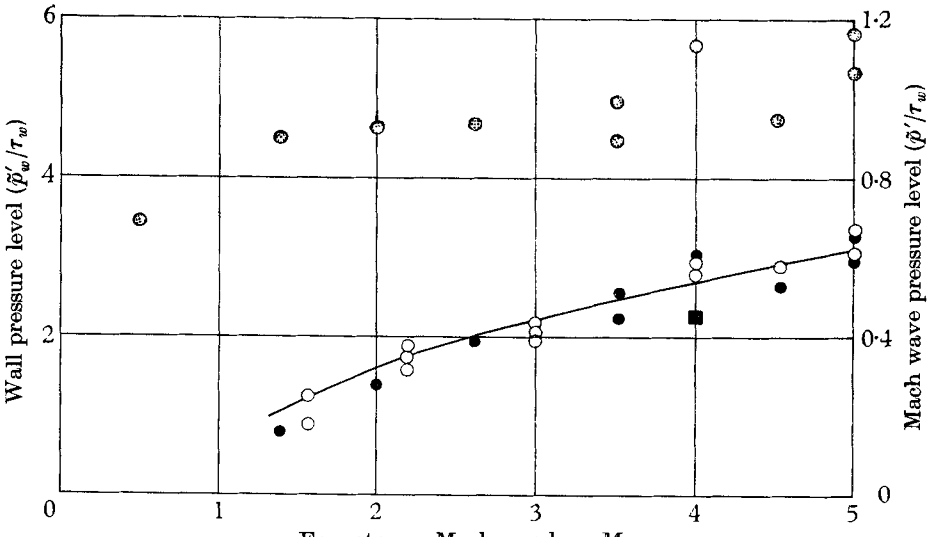

FIGURE 2. A comparison of the Mach wave strength predicted by the present theory with Laufer's experimental results (with eddy elongation). ■, Radiated pressure* (single wall configuration); ○, $\tilde{p}'/\tau_w$ radiated pressure (four-wall configuration) (Laufer 1964); ●, Mach wave strength predicted by present theory; ⊙, $\tilde{p}'_w/\tau_w$ wall pressure (Kistler & Chen 1963).

Laufer (1964) reports values of both $\tilde{p}'/\tau_w$ and $\tilde{p}'_w/\tau_w$, the latter obtained in the same experimental apparatus by Kistler & Chen (1963), and those values are reproduced here in figures 1 and 2. Superimposed on those figures are the results of this analysis, based on Kistler & Chen's (1963) surface pressure measurements. In figure 1, we neglect the effects of eddy distortion and account for them in figure 2. Although these values are factored for the best correspondence with Laufer's measurements, that factor is so close to unity that it represents a startling agreement between the theoretical and experimental values. In fact, the factor is 2·3 in figure 1 and 1·5 in figure 2. The theoretical points displayed are computed from equation (3.12) with ϵ taken as 0·2 and the right-hand side multiplied by these values.

4. Mach wave radiation from cylindrical shear layers

The apparent agreement between our theoretical estimate and Laufer's measurements in the turbulent boundary layer, leads us to speculate on the nature of the Mach wave field in situations more closely resembling a rocket exhaust flow. Those Mach waves still await a detailed study so that, in this case, our computation must be regarded as a prediction that might be checked by future experiment.

Again we choose an idealized flow of simple geometry. We assume the turbulent shear layer to surround a cylindrical region of supersonic laminar flow. The mean flow is again alined in the 1-direction and 2 denotes the radial direction. Only the simplest of cases is considered, where the observer is far enough away from the flow for both the jet diameter D and the finite length of uniform supersonic flow ΔL to be small compared with $|\mathbf{x}-\mathbf{y}|$. In addition, the shear layer thickness is assumed small in comparison with the jet diameter.

This example falls a long way short of representing real exhaust flows, but may provide a basis for computing more realistic examples by summing over all lengths ΔL. That is not attempted here since the estimate is based on the assumption that one knows the distribution of near field pressure very close to the flow as well as the axial velocity variation. Neither of these are known at the present time.

An isolated length of uniform supersonic flow may be established by shielding all but a small region of a rocket exhaust from its environment. An experiment can then be foreseen where our prediction can be checked and extensions to more realistic situations could follow should such checks prove encouraging.

Our estimate is based, as was the previous example, on equation (3.3). The integration around a line of constant mean velocity is straightforward, as is that in the downstream direction where the integrand is constant over the small length ΔL. Again we introduce the directionality of the radiation field through the function $F(M,\theta)$.

$$\overline{p^2}(\mathbf{x}) = \frac{D}{32\pi}\,\Delta L \int F(M,\theta)\,\frac{\cos^2\theta\sin^2\theta}{|\mathbf{x}-\mathbf{y}|^2}\left\{\frac{1}{a^2}\frac{\partial \overline{u}_1}{\partial y_2}\right\}^2 (y_2)\,\overline{\left(\frac{\partial p}{\partial t}\right)^2}\,(y_2)\,a_0\tau^*\lambda_s^*\,dy_2. \quad (4.1)$$

We estimate the typical pressure level based on that present in the near vicinity of the flow $\overline{p_0^2}$, and follow identical procedures to those adopted in the foregoing section, for interchanging variables and assessing the integral scales τ^* and λ_s^*. Only one point is materially different, and that concerns the distribution of temperature likely to be met in the shear layer of a rocket exhaust. The increase in temperature due to flow retardation is likely to be a small effect in comparison with the temperature gradient induced by mixing of the hot exhaust gases with the quiescent air. We shall suppose this to be the case and conjecture that the mean temperature is proportional to the mean velocity $a_0 M$. That variation will bring out at least the major effects that can be extrapolated from experiments in hot jet flows (Pai 1954) albeit at considerably lower exhaust speeds, where the temperature and velocity profiles bear a pronounced similarity.

Accordingly we write a^2 as a fraction of a_0^2, and carry out the steps that reduce equation (4.1) to the analogue of equation (3.9)

$$\frac{a^2}{a_0^2} = 1 + \frac{M}{M_\infty}\left\{\frac{T_\infty}{T_0} - 1\right\}, \tag{4.2}$$

$$\overline{p^2}(\mathbf{x}) = \frac{5D}{64\pi}\Delta L\overline{p_0^2}\int F(M,\theta)\frac{\cos^2\theta\sin^2\theta}{|\mathbf{x}-\mathbf{y}|^2}\left\{1 + \frac{M}{M_\infty}\left[\frac{T_\infty}{T_0} - 1\right]\right\}^{-2} M^3\, dM, \tag{4.3}$$

where $a_0 M_\infty$ and T_∞ are the nozzle exit velocity and temperature, and T_0 is the temperature of the quiescent air into which the jet exhausts. In current rocket systems, the ratio $M_\infty^{-1}(T_\infty/T_0 - 1)$ is close to unity, so that it may be discarded in equation (4.3)

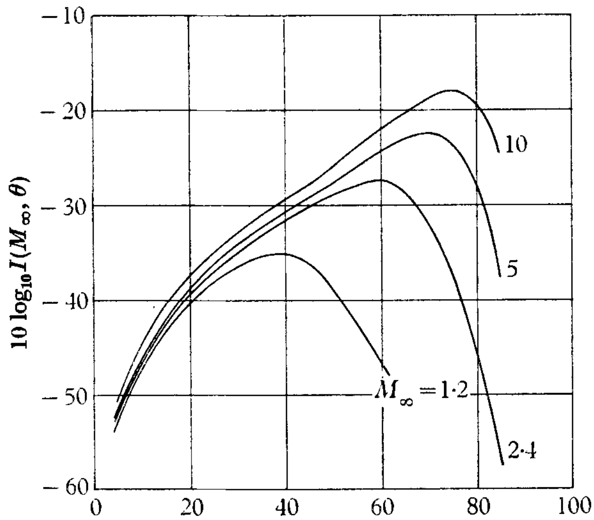

FIGURE 3. Computed values of the directionality function $I(M_\infty, \theta)$, equations (4.4) and (4.5), for Mach waves radiated from cylindrical shear flows. $a_0 M_\infty$ is the velocity of the laminar flow at the jet centre.

Our final expression then becomes a ratio of radiation pressure to near field pressure

$$\overline{p^2}(\mathbf{x}) = \frac{5\overline{p_0^2}}{64\pi}\cos^2\theta\sin^2\theta\frac{D\Delta L}{|\mathbf{x}-\mathbf{y}|^2}\int_1^{M_\infty}\frac{M^3}{(1+M)^2}\left\{\frac{\epsilon^2 M^2}{(1-M\cos\theta)^2+\epsilon^2M^2}\right\}^{\frac{5}{2}} dM. \tag{4.4}$$

This result is simplified by the introduction of a proportionality function $I(M_\infty, \theta)$ defined such that

$$\overline{p^2}(\mathbf{x}) = \frac{D\Delta L}{|\mathbf{x}-\mathbf{y}|^2} I(M_\infty, \theta)\,\overline{p_0^2}. \tag{4.5}$$

$I(M_\infty, \theta)$ now accounts for all the directional properties of the radiated sound and it is interesting to compare its form with the known directionality of quadrupoles convected uniformly at a single speed. To do this we have computed its value with $\epsilon = \frac{1}{5}$ for several different Mach numbers and angles of emission. The results are shown graphically in figure 3 and present an interesting comparison with the single-speed cases treated by Ffowcs Williams (1963). The main difference, as one might expect, is that the extreme directionality of that example has given way to a more omnidirectional distribution, but greatly affected by the $\sin^2\theta\cos^2\theta$ term that arose from the directionality of the important Mach wave source. Nevertheless, there is a tendency for the radiation to concentrate near the Mach angle associated with the highest velocity M_∞. In rocket systems, the maximum velocity would vary with downstream position so that the radiation might be described by a suitable superposition of these results, but weighted according to the value of $\overline{p_0^2}$ at each station.

It is doubtful that these results apply, in an unmodified form, to actual rocket systems. An agreement would inevitably rely on the formation of a long, slowly expanding shock-free flow, for that is the hypothetical flow system for which the theory is intended. However, one is encouraged in advocating experiments, by the fact that the results of the two-dimensional example agree so well in the boundary-layer case. It may well be that an experimental study of the Mach wave field of a high velocity jet, exhausting from a properly designed nozzle, will yield the data with which these theoretical predictions can be compared. Such data is anxiously awaited.

5. Conclusion

In summarizing this theoretical study, in which many of the steps are of a conjectural nature, it is pertinent to ask, to what degree did our assumptions regarding the turbulent structure render fortuitous the excellent agreement between theory and experiment? To clarify this point we list the major assumptions involved, and comment on the way the results might be affected by changes in our model of the flow.

The assumptions start with the selection of the only terms directly amplified by a mean velocity gradient, as the dominant Mach wave source (see equation (2.12)). This step is central to the theory and should be regarded more as a description of the class of problems for which the theory is intended, rather than as a compromise introduced for ease of solution, but of course it is that too. The more speculative assumptions concern our model of the turbulent structure. That turbulent pressure scales are directly proportional to, and of the same order of magnitude as, the shear layer thickness seems utterly reasonable, as does the conjecture that the pressure levels vary little within that scale. But to relate the eddy lifetime to the inverse of local shear is a step justified only by a distant analogy with known properties of subsonic jet mixing regions. The elongated nature of the eddies is likewise a tentative extrapolation of known features of subsonic flows. But both these steps would seem to affect only details of the result, for their influence appears negligible in comparison to that of temperature changes, accounting for a factor proportional to the fourth power of Mach number at high speeds (i.e. the factor $(1+0{\cdot}2M^2)^{-2}$ in equation (3.12)). Yet a complete neglect of the temperature term would alter the results shown in figure 1 by a factor less than $2{\cdot}5$ at $M_\infty = 5$, a point that leads us to suggest that the estimation technique we advocate is relatively insensitive to details of the flow. That feature gives us confidence that the discrepancies between our model and real flows will not materially affect the result. However, it would be wrong to underestimate the measure of coincidence that contributed to the agreement of our calculations with Laufer's results. The computation technique is coarse and based on a model flow of questionable detailed significance, so that our agreement with experiment closer than a factor of three comes as a considerable surprise. To what degree it is fortuitous will be answered far more effectively by the results of experiments in high speed jet flows, for there we have predicted the Mach wave fields in the absence of experimental data. More strictly, we have predicted the radiation field of a model flow designed to simulate part of the supersonic region of a rocket exhaust stream. It may be feasible to isolate that supersonic part, on a model scale, by exposing to the atmosphere only the initial length of mixing flow, and thereby creating an experiment where our predictions could be critically tested.

One final point concerns the asymptotic behaviour of the radiation field at high Mach number. Our equations predict that the radiated pressure increases in direct proportion to that of the near field. This point, though compatible with the idea that Mach waves have no conventional near field (so that their strength near the source is a direct measure of their radiated energy), throws no new light on the asymptotic velocity dependence at high speed. It merely transfers the emphasis to the near field pressure. The more general dimensional arguments, that have previously been applied to sound generation at supersonic speeds, indicate that the radiated energy should increase in direct proportion to the mechanical energy of the flow. Those arguments could be brought up again, so that in conjunction with the present result, they would predict the mean square pressure, in both the near and far fields of supersonic shear flows, to have a velocity cubed dependence. This result, essentially restricted to geometrically similar flows, is another point that could be checked very easily, but has yet to be studied experimentally. Such experiments are urgently needed for they seem the only way of verifying the theoretically inspired ideas that Mach waves form a significant part of the noise field of modern rockets.

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